The Blockwise Navarro Alperin Weight Conjecture for Symmetric and Alternating Groups and Their Double Covering Groups

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- Let G be a finite group and p be a prime number. Let $\mathbb{Q}_p$ be the field of p-adic numbers and $\bar{\mathbb{Q}}_p$ be its algebraic closure. Let $\mathbb{F}_p$ be the finite field of p-elements and $\bar{\mathbb{F}}_p$ be its algebraic closure. Denote by $\mathrm{Irr}(G)$ the set of irreducible ordinary characters of G and $\mathrm{IBr}_p(G)$ the set of irreducible p-Brauer characters of G.
- Let R be a p-subgroup of G and $N_G(R)$ be the normalizer of R. Let φ be an irreducible p-defect-zero character of $N_G(R)/R$, that is, $\varphi(1)_p = |N_G(R)/R|_p$. The pair (R, φ) is called a p-weight of G. Denote by $\mathcal{W}_p(G)$ the set of conjugacy classes of p-weights (R, φ) of G.

- Let B be a p-block of G.
- Let $\mathrm{IBr}_p(B)$ be the set of irreducible p-Brauer characters in block B.
- If (R, φ) is a p-weight of G, view φ as an irreducible ordinary character of $N_G(R)$ by inflation and let b be the block of $N_G(R)$ containing φ. Then (R, φ) is called a p-weight in B if $b^G = B$. Let $\mathcal{W}_p(B)$ be the set of conjugacy classes of p-weights in B.

Conjecture (Blockwise Alperin Weight Conjecture)

Let G be a finite group, p be a prime number, and B be a p-block of G. Then we have

$$|\mathrm{IBr}_p(B)| = |\mathcal{W}_p(B)|.$$

Navarro Alperin Weight Conjecture

In 2004, Navarro proposed the Galois refinement of several global-local conjectures in [1], including the Alperin weight conjecture.

Let ω be a primitive $|G|$-th root of unity.

Conjecture (Blockwise Navarro Alperin Weight Conjecture)

Let G be a finite group, p be a prime number. Then there is a $\mathrm{Gal}(\mathbb{Q}_p(\omega)/\mathbb{Q}_p)$-equivariant bijection

$$\Omega\colon \mathrm{IBr}_p(G) \to \mathcal{W}_p(G)$$

such that φ and $\Omega(\varphi)$ are in the same block for any $\varphi \in \mathrm{IBr}_p(G)$.

G. Navarro, The McKay conjecture and Galois automorphisms. *Ann. of Math.* **160**:3 (2004), 1129–1140.

Navarro Alperin Weight Conjecture

Theorem (A. Turull, [1], 2014)

Let p be a prime number, G be any p-solvable group, and Q be a p-subgroup of G. Let U be a normal p'-subgroup of G and let $V = C_U(Q)$. Suppose $\zeta \in \mathrm{Irr}(U)$ is Q-invariant, $\xi \in \mathrm{Irr}(V)$ is the Glauberman correspondent of ζ. Then there exists a bijection

$$f : \mathrm{IBr}_p(G, Q \mid \zeta, \mathbb{Q}_p) \to \mathcal{W}_p(G, Q \mid \xi, \mathbb{Q}_p).$$

Let $\psi \in \mathrm{IBr}_p(G, Q \mid \zeta, \mathbb{Q}_p)$ and $f(\psi) = \phi \in \mathcal{W}_p(G, Q \mid \xi, \mathbb{Q}_p)$.

1. $\mathrm{bl}(\psi) = \mathrm{bl}(\phi)^G$;
2. $\mathrm{codeg}(\psi)^2_{p'} \equiv \mathrm{codeg}(\phi)^2_{p'} \mod p$;
3. *f commutes with the action of $\mathrm{Gal}(\mathbb{Q}_p(\omega)/\mathbb{Q}_p)$.*

A. Turull, The strengthened Alperin weight conjecture for p-solvable groups, *J. Algebra* **398** (2014), 469-480.

The symmetric group S_n is the Weyl group of A_{n-1} type, that is,

$$S_n = \langle \bar{s}_i, 1 \leq i \leq n-1 \mid \bar{s}_i^2 = (\bar{s}_i\bar{s}_{i+1})^3 = 1, [\bar{s}_i, \bar{s}_j] = 1, |i-j| \geq 2 \rangle.$$

There are two non-isomorphic central extensions of S_n by $\{\pm 1\}$, which is called double covering groups of S_n, defined as

$$\tilde{S}_n^+ = \langle s_i, 1 \leq i \leq n-1 \mid s_i^2 = (s_is_{i+1})^3 = 1, [s_i, s_j] = -1, |i-j| \geq 2 \rangle;$$
$$\tilde{S}_n^- = \langle s_i, 1 \leq i \leq n-1 \mid s_i^2 = (s_is_{i+1})^3 = -1, [s_i, s_j] = -1, |i-j| \geq 2 \rangle.$$

The kernel of the sign character is called the double covering group of A_n, denoted as $\tilde{A}_n$.

Main Results

Theorem (Y. D.–X. Huang, [1], 2025)

The blockwise Navarro Alperin weight conjecture holds for S_n, A_n.

Theorem (Y. D.–X. Huang–J. Zhang, [2], 2026+)

The blockwise Navarro Alperin weight conjecture holds for $\tilde{S}_n^{\pm}$, $\tilde{A}_n$.

Moreover, these bijections are also equivariant under the action of automorphism groups and linear character groups.

Y. D., X. Huang, The blockwise Galois Alperin Weight Conjecture for symmetric and alternating groups, *Math. Z.* **311** (2025).

Y. D., X. Huang, J. Zhang, The blockwise Navarro Alperin weight conjecture for covering groups of symmetric groups and alternating groups, arXiv:2509.13673.

- Let ω_m be a primitive m-th root of unity.
- Let $\sigma \in \mathrm{Gal}(\mathbb{Q}_p(\omega_{2n!})/\mathbb{Q}_p)$ such that $\sigma(\xi) = \xi^p$ for any p'-th root of unity ξ.
- For any $\beta \in \mathrm{IBr}_p(S_n)$, we have $\mathbb{Q}(\beta) = \mathbb{Q}$.
- For any p-weight (R, φ) of S_n, we have $\mathbb{Q}(\varphi) = \mathbb{Q}(\omega_{p-1})$.
- Thus all irreducible p-Brauer characters and p-weights are invariant under the action of σ.

For odd primes:

- Both irreducible p-Brauer characters and conjugacy classes of p-weights in block B_κ are parametrized by

$$\Lambda_\kappa = \{\lambda \vdash n \mid \lambda_{(p)} = \kappa, \lambda_{(p-1)/2} = \emptyset\}.$$

- If λ is non-self-associated, then λ and λ' are counted once. Both of irreducible p-Brauer characters and p-weights are invariant under the action of σ.
- If λ is self-associated, then λ is counted twice. The action of σ is determined by

$$\nu_{\kappa,w} = \nu_\kappa \cdot \left(\frac{-1}{p}\right)^{w/2}.$$

Here $n = |\kappa| + pw$.

For Alternating Groups

For prime 2:

- Both irreducible p-Brauer characters and conjugacy classes of p-weights in block B_κ are parametrized by partitions of w. Here $n = |\kappa| + 2w$.
- If an irreducible 2-Brauer character or 2-weight is invariant under the action of S_n, then it is also invariant under the action of σ.
- Otherwise $2 \mid w$ and the action of σ is determined by

$$\nu_{\kappa,w} = \nu_\kappa \cdot (-1)^{w/2}.$$

It suffices to consider spin blocks of odd primes.
Both irreducible p-Brauer characters and conjugacy classes of p-weights in spin block B_κ are parametrized by

$$\Lambda_\kappa = \{\lambda \vdash n \text{ strict} \mid \lambda_{(p)} = \kappa, \lambda_0 = \emptyset\}.$$

Define

$$\nu_{\kappa,w} = \begin{cases} \nu_\kappa \cdot \left(\frac{-1}{p}\right)^{\lfloor w/2 \rfloor} \left(\frac{2\eta}{p}\right)^w, & \text{if } \kappa \text{ is non-self-associated,} \\ \nu_\kappa \cdot \left(\frac{-1}{p}\right)^{\lceil w/2 \rceil} \left(\frac{2\eta}{p}\right)^w, & \text{if } \kappa \text{ is self-associated.} \end{cases}$$

Here $\eta = +1$ for $\tilde{S}_n^+$ and $\eta = -1$ for $\tilde{S}_n^-$.

For Double Covers

- If λ is non-self-associated, then
  - for $\tilde{S}_n^{\pm}$, λ is counted twice and the action of σ is determined by $\nu_{\kappa,w}$;
  - for $\tilde{A}_n$, λ is counted once and both irreducible p-Brauer characters and p-weights are invariant under the action of σ.
- If λ is self-associated, then
  - for $\tilde{S}_n^{\pm}$, λ is counted once and both irreducible p-Brauer characters and p-weights are invariant under the action of σ;
  - for $\tilde{A}_n$, λ is counted twice and the action of σ is determined by $\nu_{\kappa,w}$.

Conjecture (The Refined Borué Conjecture)

For an arbitrary complete discrete valuation ring $\mathcal{O}$ and a block idempotent b of a finite group G over $\mathcal{O}$ with an abelian defect group, there is a splendid Rickard equivalence between $\mathcal{O}Gb$ and its Brauer correspondent.

Theorem (Y. D.–X. Huang, [1], 2026+)

The refined Broué conjecture holds for spin RoCK blocks of $\tilde{S}_n^{\pm}$ and $\tilde{A}_n$ and odd primes.

Y. D., X. Huang, On RoCK blocks of double covers of symmetric and alternating groups and the refined Broué conjecture, arXiv:2510.02147.

Thank You!